

Energy-based model of forming subgroups on finite metric space

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ABSTRACT

Local interactions between particles of a collection causes all particles to reorganize in new positions. The purpose of this paper is to construct an energy-based model of self-organizing subgroups, which describes the behavior of singular local moves of a particle. The present paper extends the Hegselmann-Krause model on consensus dynamics[12], where agents simultaneously move to the barycenter of all agents in an epsilon neighborhood. The Energy-based model presented here is analyzed and simulated on finite metric space.

Key Words: *Group dynamics, self-organizing groups; population dynamics; collective intelligence; forming groups; metric on finite sets.*

AMS Subject Classifications:81T80; 93A30; 37M05; 68U20

1 Introduction

In real life are many phenomena of grouping, self-organizing or reorganizing groups. In systems biology for example, local interactions, reactions and transformations of cells have complex structural and natural backgrounds. That one cell becomes a brain cell and another a liver cell may depend on its ambient: for example the pressure from faster growing cells on top of one cell may cause it to become a brain cell. Thus, the fate of a single cell seems to be largely at random, whereas the result: the human being is very well defined stable and obviously fixed in advance[7, 8, 9]. Motivated by the importance of biological phenomena in real life, we present a thorough analysis of the behavior of cell interactions in population models characterized by heterogeneous connectivity and mobility patterns, we observe particles, which are permanently moving to position with minimal energy. These local interactions of a collection of particles considered as local rule causes all particles to reorganize in new positions. The purpose of this paper is to construct a model, which describes such phenomena in finite metric space. Similarly, the opinion dynamic modeling is recently presented by Hegselmann and Krause in [12]. This consists of simultaneous movements of all elements of the groups. The so called agents change his position by taking the average of all positions on his epsilon neighborhood. The fact that all agents are reacting simultaneously can be understood as a central instruction. Therefore, we propose a model based on the local energy without centralization, i.e. Synchrony and asynchrony reaction are possible. Each member of the group is permanently looking for better position following a gradient trajectory or the principle of barycenter on his

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neighborhood. Such phenomena are studied by many scientists in other contexts as schooling or flocking [2, 3, 5, 10, 11].

We are interested to extend the idea of HK-model to an energy based model of condensing sequences. Therefore, we denote the collections of particles by a positive measure m and consider that each mass points is permanently changing his spatial position in an arbitrary finite metric space with metric d , by looking for positions, where the local energy with respect to m :

$$e_\varepsilon(a, m) = \int_{d(a, y) \leq \varepsilon} d^2(a, y) m(dy), \quad (1)$$

for a positive confidence ε , is minimal.

The energy based model presented here is a non linear dynamical system on a metric space X . Which, in this paper, it will simulated only in discrete time on a finite metric space. A particle changes his position wherein his energy is smaller. Thus, our theorem says that, even if this rule is local, it has an influence of the global energy of the measure m :

$$E_\varepsilon(m) = \int_X e_\varepsilon(x, m) m(dx), \quad (2)$$

It is important to note that, the arbitrary range of the reactions of the particles and the non uniqueness of the positions minimizing the local energy give a source of stochastic investigations, which we analyze by a future work.

Our goals are, to develop a new model of condensing sequences of positive measures with some theoretical analysis. We deal only with a discrete measure on a discrete metric space, while our attention is focused also to extend such a model to a continuous metric space in a working paper. This paper is structured in two principal sections. The first one proposes the construction of condensing sequences. The second section proposes a numerical simulation of such a phenomena. Some general remarks are listed.

2 Condensing model on finite metric space

Let (X, d) be a finite metric space with metric d . A non negative measure m on X is represented by a function $m : X \rightarrow [0, \infty)$ in the obvious way. Denote by $M_+(X)$ the set of all positive measures on X . A measure $m \in M_+(X)$ is given as

$$m = \sum_{x \in X} m(x) \delta_x = \sum_{x \in S(m)} m(x) \delta_x,$$

where δ_x denotes the Kronecker symbol and by $S(m)$ we denote the support of m given as $S(m) := \{y \in X | m(y) > 0\}$. Fix a real number $\varepsilon > 0$. The ε -energy of m is

$$E_\varepsilon(m) = \sum_{d(x, y) \leq \varepsilon} m(x) m(y) d^2(x, y). \quad (3)$$

The ε - energy of point $a \in X$ with respect to m is

$$e_\varepsilon(a, m) = \sum_{d(a, y) \leq \varepsilon} m(y) d^2(a, y). \quad (4)$$

Let a pair $(a, a^*) \in X \times X$ operates on the set $M_+(X)$ of nonnegative measures as $m \rightarrow (a, a^*, m)$, where

$$m^* := (a, a^*, m)(x) := \begin{cases} m(x); & \text{if } x \notin \{a, a^*\}, \\ 0; & \text{if } x = a, \\ m(a) + m(a^*); & \text{if } x = a^*. \end{cases}$$

The mapping above is a mass translating map. Where, the move of a to a^* means that the mass of a^* will adjusted by a new mass, namely the mass of a . In this context, we define a new global mass denoted by m^* and in order to observe this influence, we propose the following example:

Example: Let us consider a three points metric space as subset of the real line. We assume that two neighbors points have a distance of one and we define a measure m by the masses punted in the three points. We denote the mass of m in each point by the the numbers given on the figures 1, 2 an 3 . For $\varepsilon = 1$ let us consider three moves, namely (a), (b) and (c) Note that

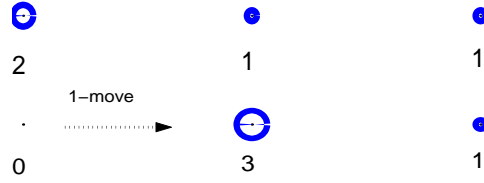


Figure 1: *Examples of moves (a).*

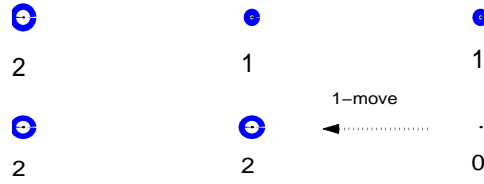


Figure 2: *Examples of moves (b).*

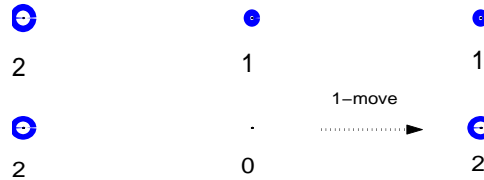


Figure 3: *Examples of moves (c).*

$$m = \sum_{i=1}^3 m(i)\delta_i, \quad m_a^* = 3\delta_2 + \delta_3, \quad m_b^* = 2\delta_1 + 2\delta_2, \quad m_c^* = 2\delta_1 + 2\delta_3.$$

and

$$E(m) = 6, \quad E(m_a^*) = 6 \quad E(m_b^*) = 8 \quad E(m_c^*) = 0.$$

We observe that,

- (a) m is transformed to m_a^* with $E(m) = E(m_a^*) = 6$.
- (b) m is transformed to m_b^* with $E(m) = 6 < E(m_b^*) = 8$.
- (c) m is transformed to m_c^* with $E(m) = 6 > E(m_c^*) = 0$.

The figures 1– 3 presents three examples of moves: (a) the mass 2 moves to the mass 1 on the middle, the total energy don't changes $E(m) = E(m_a^*) = 6$. (b): The mass 1 on the left moves to the mass one on the middle, we remark that the total energy increases $E(m_b^*) = 8$. (c) it shows that the move of the middle mass 1 to the right one 1 causes a vanishing of the total energy. Therefore, we are looking for a local rule for a move of particles, which causes an increasing of the global energy.

Definition 2.1 (The model). *A pair (m, m^*) is called an ε –move, if there is a pair $(a, a^*) \in X \times X$ such that:*

- (i) $m^* = (a, a^*, m)$,
- (ii) $d(a, a^*) \leq \varepsilon$.
- (iii) $e_\varepsilon(a, m) > e_\varepsilon(a^*, m^*)$.

To illustrate this, we presents the following figure of a move based on the energy, where the particle has only one neighbors in the new position: We are interested in sequences of non

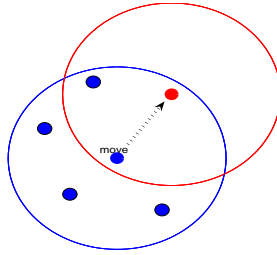


Figure 4: *Examples of moves based on the energy.*

negative measures m^1, m^2, \dots satisfying the conditions above. Such a sequence m^1, m^2, \dots is called ε –condensing. Clearly for every $a, a^* \in S(m)$ if $d(a, a^*) \leq \varepsilon$, then either (a, a^*, m) or (a^*, a, m) is an ε –move. Therefore, whenever $E_\varepsilon(m) > 0$ there is an ε –move (m, m^*) . Thus, for every finite m with non vanishing energy, there is an ε –condensing sequence m^1, m^2, \dots . Our theorem says that such a sequence is finite.

The simultaneous displacement sequences are studied in another context in literature by synchronous communication, moves and reactions, for example, we refer to the models studied in [12, 13]. The resulting measure of a condensing sequence depends not only on the initial measure, but also on the order of succession of the particles reactions. Hence, we introduce a random range of the order of reaction of particles. Which give a source of stochastic investigations. The same idea was proposed and developed by Sieveking [13] in the case of the real line. The purpose of the following is to prove the convergence of ε -condensing sequences:

Theorem 2.1. *A singularly ε -condensing sequence is finite.*

The proof will be a consequence of the following lemmas.

Lemma 2.1. *Let $m \in M_+(X)$, $a, a^* \in X$ such that $d(a, a^*) \leq \varepsilon$. Then*

$$E_\varepsilon(m) - E_\varepsilon(m^*) = 2m(a)[e_\varepsilon(a, m) - e_\varepsilon(a^*, m) + m(a)d^2(a, a^*)], \quad (5)$$

where $m^* = (a, a^*, m)$.

Proof: To simplify, we use the following notation

$$I_m := \sum_{d(x,y) \leq \varepsilon; \{x,y\} \cap \{a,a^*\} = \emptyset} m(x)m(y)d(x,y)^2 \quad (6)$$

Let us compute the energy of m :

$$\begin{aligned} E_\varepsilon(m) &= \sum_{d(x,y) \leq \varepsilon} m(x)m(y)d^2(x,y) \\ &= I_m + 2m(a) \sum_{d(a,x) \leq \varepsilon} m(y)d^2(x,y) + 2m(a^*) \sum_{d(a^*,y) \leq \varepsilon} m(y)d^2(a^*,y) \\ &\quad - 2m(a)m(a^*)d^2(a, a^*) \end{aligned}$$

then

$$\begin{aligned} E_\varepsilon(m) &= I_m + 2m(a)e_\varepsilon(a, m) + 2m(a^*)e_\varepsilon(a^*, m) \\ &\quad - 2m(a)m(a^*)d^2(a, a^*). \end{aligned} \quad (7)$$

Similarly for $m^* = (a, a^*, m)$:

$$E_\varepsilon(m^*) = I_{m^*} + 2m^*(a)e_\varepsilon(a, m^*) + 2m^*(a^*)e_\varepsilon(a^*, m^*) - 2m^*(a)m^*(a^*)d^2(a, a^*)$$

Note that

$$I_m = I_{m^*}; \quad m(a) = 0; \quad m^*(a^*) = m(a) + m(a^*).$$

and

$$e(a^*, m^*) = e(a^*, m) - m(a)d^2(a^*, a). \quad (8)$$

Therefore

$$\begin{aligned} E_\varepsilon(m^*) &= I_m + 2(m(a) + m(a^*))e_\varepsilon(a^*, m^*) \\ &= I_m + 2(m(a) + m(a^*))(e(a^*, m) - m(a)d^2(a^*, a)). \end{aligned} \quad (9)$$

and from (7) and (9) it follows:

$$E_\varepsilon(m) - E_\varepsilon(m^*) = 2m(a)[e_\varepsilon(a, m) - e_\varepsilon(a^*, m) + m(a)d^2(a, a^*)]. \quad (10)$$

Lemma 2.2. For $m \in M_+(X)$ let $n(m)$ be the number of elements $a \in X$ such that $m(a) > 0$. If m^1, m^2, \dots is a sequence of measures on X which is singular and ε -condensing, then

1. $i \rightarrow E_\varepsilon(m^i)$ is strictly decreasing
2. $i \rightarrow n(m^i)$ is decreasing

Proof: The first claim follows from lemma 2.1. To show the second let $S(m) = \{x \in X \mid m(x) > 0\}$ be the support of the measure m . Consider $m^* = (a, a^*, m)$. If $a \notin S(m)$ then $S(m) = S(m^*)$ and $n(m) = n(m^*)$. If $a \in S(m)$ and $a^* \in S(m)$ then $S(m^*) = S(m) \setminus \{a\}$ and $n(m^*) < n(m)$. If $a \in S(m)$, $a^* \notin S(m)$ then

$$S(m^*) = (S(m) \setminus \{a\}) \cup \{a^*\},$$

and again $n(m) = n(m^*)$.

Proof of theorem 2.1. Let m^1, m^2, \dots be an infinite sequence of measures, which is ε -condensing. Because of the preceding lemma, we may assume that $i \rightarrow n(m^i)$ is constant. Hence, for every i The measure m^{i+1} is a permutation of m^i i.e. $m^{i+1} = m^i \circ \pi_i$, where $\pi_i : X \rightarrow X$ is a permutation of X . Therefore, $m^i = m^1 \circ \pi_1 \circ \dots \circ \pi_{i-1}$. As the group of permutations of X is finite, there exist a natural numbers $i, k > 0$ such that

$$\pi^1 \circ \dots \circ \pi^i = \pi^1 \circ \dots \circ \pi^{i+k}, \quad (11)$$

and $m^{i+1} = m^{i+k+1}$, which however is impossible in view of the second claim of the previous lemma.

There exist infinite non converging condensing sequences: Consider a simultaneously condensing sequence with two mass points $m_n = m_s$, where m_n is the mass of a point in the north pole of unit circle and m_s is a mass of a point in the south pole. Note that, this metric space is not a finite metric space but to explain this example in a finite metric space, one can use only four points metric spaces, namely the north, the south pole and the midpoints of them on the unit circle. Here, m is given as $m := m_s \delta_{\frac{3\pi}{2}} + m_n \delta_{\frac{\pi}{2}}$. If we consider the rule of simultaneously moves (HK-model) studied by [12]. An admissible moves scenario is the periodic one, namely $\frac{3\pi}{2}$ moves to 0 and $\frac{\pi}{2}$ moves to π . The condensing sequence constructed above m^1, m^2, \dots is simultaneously condensing and does not converge. We can also construct another type of non converging condensing sequences. We believe that, in this case, non converging sequences have a periodic behavior. In the case of the existing of many positions minimizing the energy, the particle moves to one of them.

3 Numerical simulations of condensing sequences

In our simulations, we do three numerical experiments on finite metric spaces as a subset of an Euclidean space. The finite set will be constructed as n points metric space and a subset of a continuous metric space. The numerical simulations are listed as follows: (a) and (b) Uniform mass distribution, (c) Uniform random mass distribution in $[0, 4]$. Let us at first define a finite metric space of 121 points (X, d) as:

$$\begin{aligned} X &= \{x_1, \dots, x_{121}\} \subset \mathbb{R}^2, \\ d(x_i, x_j) &= \|x_i - x_j\|_2, \forall i, j = 1, \dots, 121, \end{aligned} \quad (12)$$

where X is a subset of \mathbb{R}^2 , the metric used here is the Euclidean one. Note also that the metric space is a uniform gridpoints of the domain $[0, 1]^2$. The initial measure will be defined as a positive measure $m := \sum_{x \in X} m(x) \delta_x$, where $S(m) = X$ and $m(x) > 0$. We run our code after fixing a random order of reactions (the array of 121 index will be permuted randomly at each iteration step). It is important to note that the positions, which minimize the energy are not unique, therefore, we choose randomly one of them. Note that the uniform random distribution generates real random numbers between 0 and 4. It is important to illustrate the numerical procedure of a move, which we have implemented: Let us suppose that a moves to a^* the question is, how will a^* be chosen from a :

1. The mass point a localizes all its neighbors positions $N_\varepsilon(a) := \{b_1, \dots, b_k, c_1, \dots, c_t\}$, such that: $t + k \leq \#X$, $m(b_i) > 0$ and $m(c_j) = 0$ for $i = 1, \dots, k$ and $j = 1, \dots, t$.
2. The energy will be computed (approximated) in $x \in N_\varepsilon(a)$, let $N_\varepsilon^*(a)$ be all neighbors position of a such that $e_\varepsilon(x, m) < e_\varepsilon(y, m)$, for all $x \in N_\varepsilon^*(a)$ and $y \in N_\varepsilon(a) \setminus N_\varepsilon^*(a)$.
3. Since $e_\varepsilon(x, m) = e_\varepsilon(y, m)$, for all $x, y \in N_\varepsilon^*(a)$, a^* will be randomly chosen by a .

The steps 1-3 are done until convergence by all mass points with positive mass and its also important to note that by a move all point of the space are considered, namely with positive or zero mass.

Our main concern here is to observe the condensing behavior of the limit state of each simulation. The figure 5 presents three condensing iterations in X of the three simulations (left, middle and right columns). The small dark dots represent the metric space and the large ones represent the particles. The initial measure is a collection of point masses such that each point of the grid has a positive mass. A move is only admissible on the small points (FMS). In this case the limit measure is a collection of ε isolated mass points. It is important to note, that the limits are attained after different number of iteration as indicated by 3. It is also important to note that, this plot, shows in the first four rows, only the center of mass of each point mass, the weight is given as a density in the last plot of figure 5. The last figure presents also the new repartition and the density of the particles at the limit state. The figures 6 show the vanishing curves of the energy of the simulation (a), (b) and (c). The following table summarizes the results of the simulations on the Euclidean finite metric space:

Parameter/Sim.	(a)	(b)	(c)
NP	121	121	121
ε	0.19	0.19	0.19
Initial state	121 masses (one)	121 masses (one)	121 masses (in U(0,4))
Final state	21 isolated masses	27 iso. masses	21 iso. masses
Time in Sec	190	168	262
Nb of iter.	290	273	215

Table 1: Results of simulations (a), (b) and (c).

Note that if m is a limit measure of a condensing sequence, then $E_\varepsilon(m) = 0$, is equivalent either to $m(X) = m(a)$ for $a \in X$ or $d(x, y) > \varepsilon$ for all $x, y \in S(m)$. Note also both cases

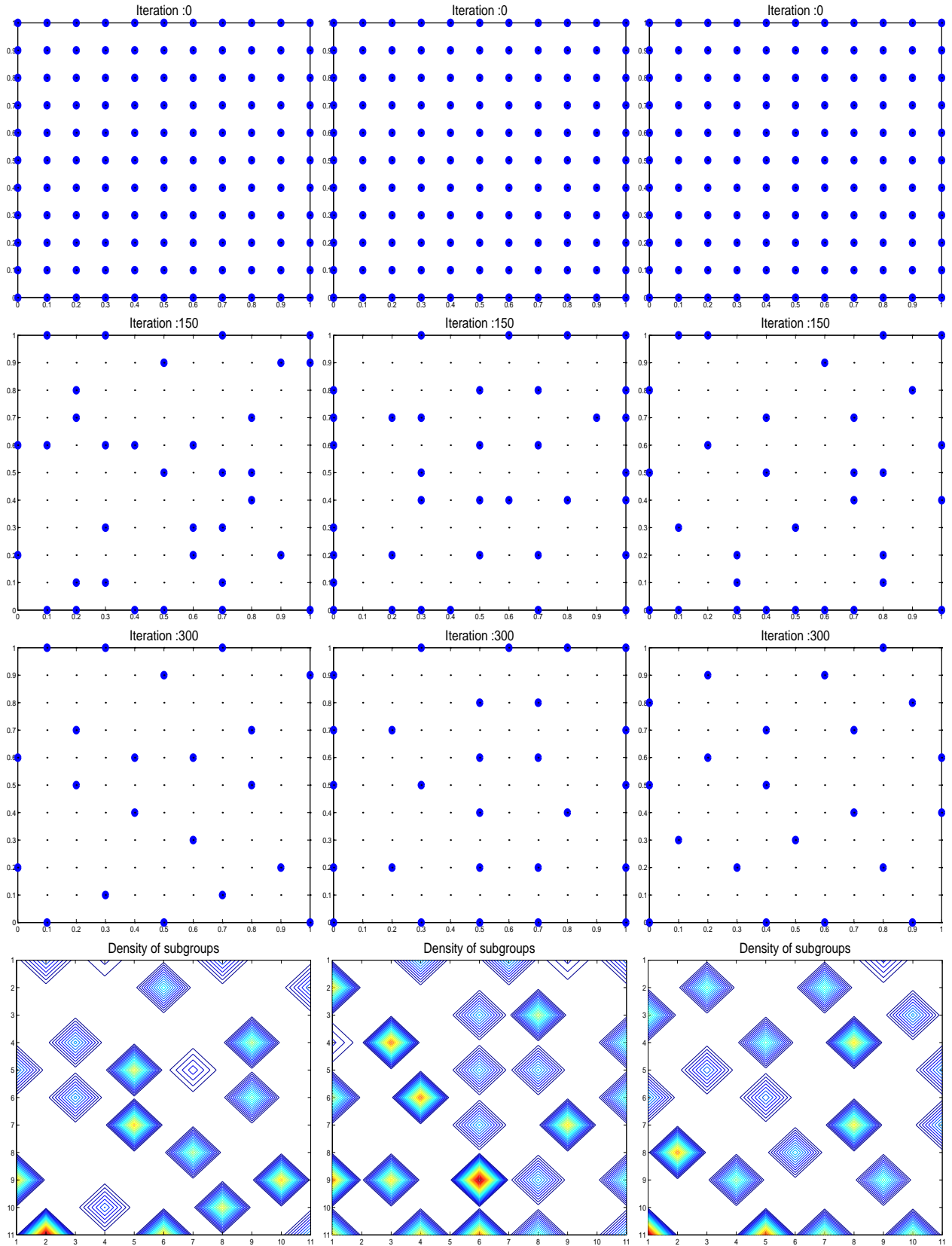


Figure 5: Condensing in an euclidian finite metric space, simulation (a) (left column), (b) (middle column), and (c) (right column).

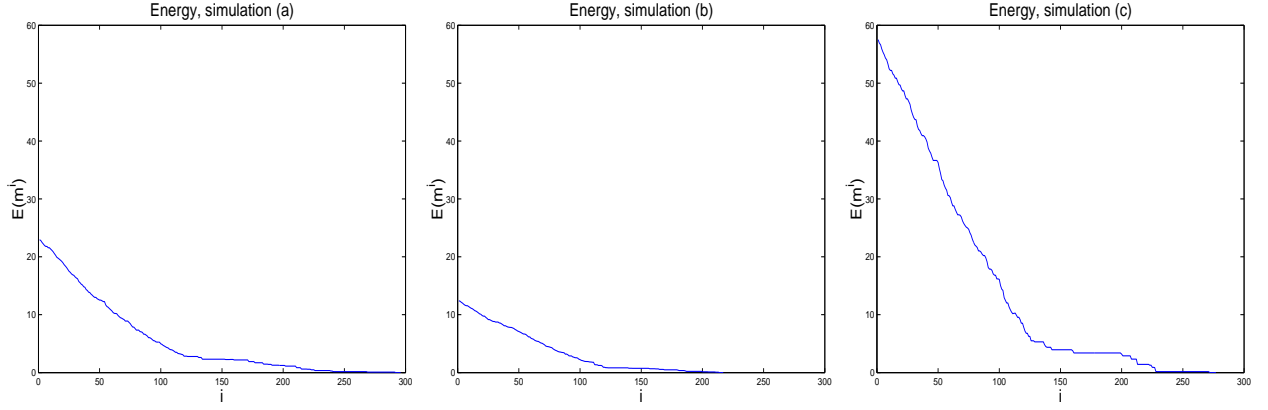


Figure 6: *Energy functions of simulation (a), (b), and (c).*

despond not only on the choice of ε but also of the indeterminately of the reactions and the non uniqueness of the points minimizing of the energy function. It is also important to note that if $\varepsilon \geq \text{diam}(X)$, then $\lim_i m^i = m(X)\delta_a$ for $a \in X$. In this case we have a total collision of the particles.

4 Concluding remarks

The present work proposes a new model for condensing sequences, with special interest on the condensing process of particles. We have observed that the limit states of the simulations (a), (b) and (c) have non uniform and different distribution of mass. They form ε isolated subgroups. In one hand, we have shown how a collection of particles with a local control rule, forms an isolated distribution of masses with zero global energy. In the other hand, we have seen, that the energy as local rule is in reality a global criteria for forming subgroups. However, one can easily show that the dynamic of the group is a consequence of individual moves of agents. Our main concern is to extend this idea to a continuous metric space, which we will be publish separately. The present study can only be considered as example for explaining the concept of consensus and emergence phenomena. It should be stressed that the stochastic behavior of our simulations is due to random choice of positions minimizing the local energy and the random range of reactions of the particles.

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